Fluid Mechanics

Chapter 1 : Similitude

Mathieu Bardoux

IUT du Littoral Côte d'Opale
Département Génie Thermique et Énergie

2nd year

Summary

Similitude model

a powerful tool for engineering

- Testing of a design prior to building $\Rightarrow$ small scale model
- Validation of theoretical models

Similitude model

a powerful tool for engineering

- Testing of a design prior to building ⇒ small scale model
- Validation of theoretical models
- Fluid mechanics ⇒ complex fluid dynamics problems ⇒ lack of reliable calculations or computer simulations

Summary

Dimensional analysis : for what purpose ?

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$\Rightarrow$ search for homogeneous relations

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Compare experiences conducted under different conditions

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Find relations between measurable quantities in various physical phenomena
$\Rightarrow$ search for homogeneous relations

Compare experiences conducted under different conditions
$\Rightarrow$ group variables into dimensionless numbers

What is a dimension ?

Common language :

Dimension ≈ Size

My car has huge dimensions

What is a dimension ?

Common language :

Dimension $\approx$ Size

My car has huge dimensions

Physicist language :

Dimension $\neq$ Size

What is a dimension ?

Common language :

Dimension ≈ Size

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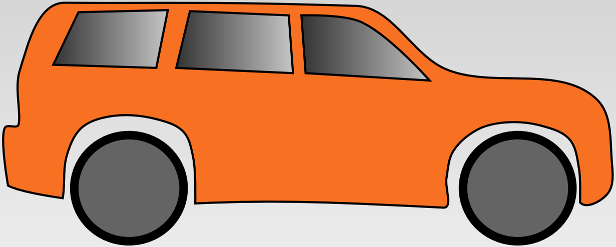
Physicist language :

Dimension ≠ Size

Dimension ≈ Nature of a physical quantity

What is a dimension ?



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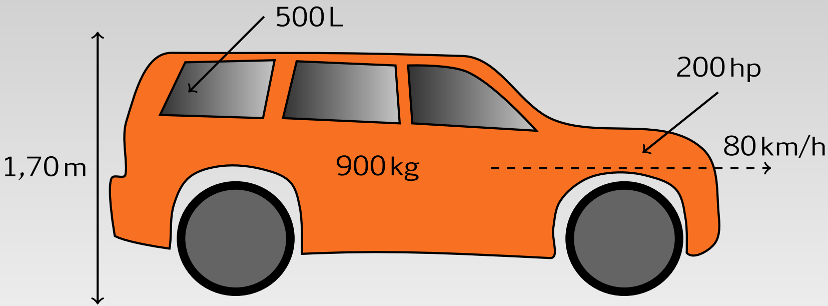
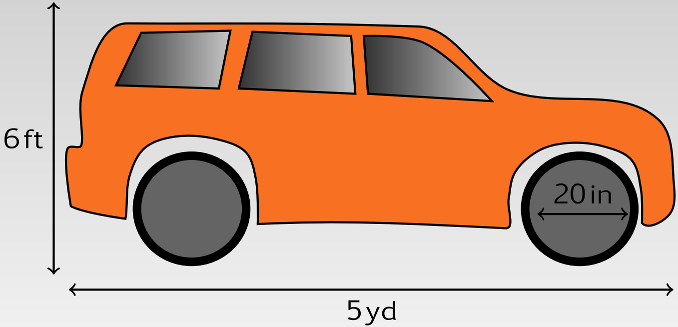


Some dimensions : length, volume, mass, power, speed. . .

Dimension ≠ unit



Length units: meter, inch, yard, parsec, toise, miles, kilometers...

Dimensional analysis : basics

- In physics, each measurable quantity is associated to a *dimension* (which is distinct from the unit in which it is given)

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Dimensional analysis : basics

- In physics, each measurable quantity is associated to a *dimension* (which is distinct from the unit in which it is given)
- Every equation must respect dimensional homogeneity
- Every physical quantity derive from a couple of *base quantities*

Rayleigh's method of dimensional analysis
Principle

Let's say a physical phenomenon: what is the law that governs it?

Rayleigh's method of dimensional analysis

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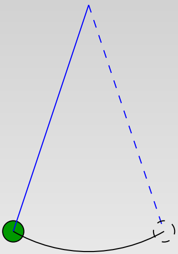
- combine them in a dimensionaly homogeneous equation

Rayleigh's method of dimensional analysis

Simple gravity pendulum



What is the value of T ?

Rayleigh's method of dimensional analysis

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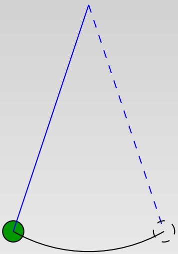

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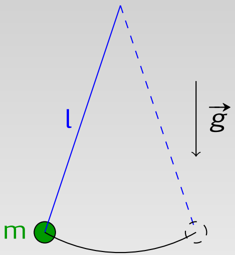

What is the value of T ?

- draw up an inventory of all the independent variables involved in the phenomenon

Rayleigh's method of dimensional analysis

Simple gravity pendulum

- draw up an inventory of all the independent variables involved in the phenomenon

$$m, l, g$$

- write the (algebraic) law as a product of these variables to a certain power
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RAYLEIGH's method of dimensional analysis

Simple gravity pendulum

- draw up an inventory of all the independent variables involved in the phenomenon
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$$T = m^{\alpha} l^{\beta} g^{\gamma}$$

- write the dimensions of all these variables as a combination of base quantities
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Simple gravity pendulum

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$$[T] = T \quad [m] = M \quad [l] = L \quad [g] = L \cdot T^{-2}$$

- combine them in a dimensionaly homogeneous equation

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$$\alpha = 0, \gamma = -\frac{1}{2}, \beta = \frac{1}{2}$$

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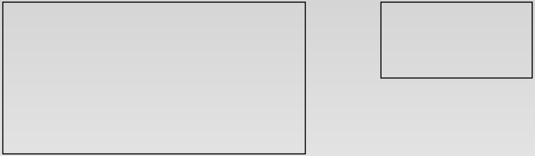
$$T = \sqrt{\frac{l}{g}}$$

Dimensionless quantities

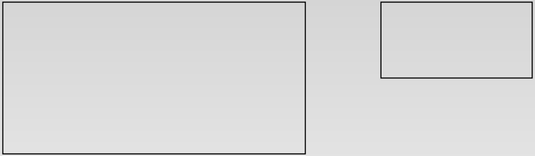
Concept of invariance



These rectangles are of different sizes

Dimensionless quantities

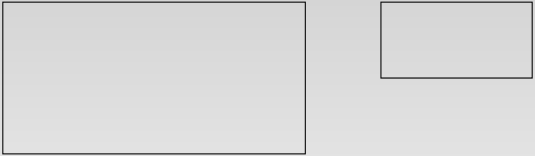
Concept of invariance



These rectangles are identical

Dimensionless quantities

Concept of invariance



These rectangles are of different sizes, yet identical

- Lenght (dimensional) change
- Proportions (dimensionless) remain the same

Buckingham π theorem

Statement

Theorem

- *If there is a physically meaningful equation involving a certain number n of physical variables, depending on k base quantities, then the original equation can be rewritten in terms of a set of $p = n - k$ dimensionless parameters $\pi_1, \pi_2, \ldots, \pi_p$ constructed from the original variables.*
- $f(x_1, x_2, \ldots x_n) = 0$ *becomes :*

$$\phi(\pi_1, \pi_2, \ldots \pi_p) = 0$$

where $\pi_1, \pi_2, \ldots, \pi_p$ are dimensionless independant parameters, functions of the variables $(x_1, x_2, \ldots x_n)$.

The number of variables has therefore decreased by k, simplifying the study of the phenomenon.

Buckingham π theorem

Application : drag force

A sphere in movement through a viscous fluid :

- Variables : D, v, ρ, μ, F
- 5-variable equation : $f(D, v, \rho, \mu, F) = 0$
- $n = 5$ variables dépending on $k = 3$ base quantities
- There is a relation of $p = n - k = 2$ variables :

$$\phi(\pi_1, \pi_2) = 0$$

Find π_1 and π_2

Buckingham π theorem

Application : drag force

Dimensional variables in the initial equation :

Buckingham π theorem

Application : drag force

Dimensional variables in the initial equation :

$$D = L$$
$$v = L \cdot T^{-1}$$
$$\rho = M \cdot L^{-3}$$
$$\mu = M \cdot L^{-1} \cdot T^{-1}$$
$$F = M \cdot L \cdot T^{-2}$$

Buckingham π theorem

Application : drag force

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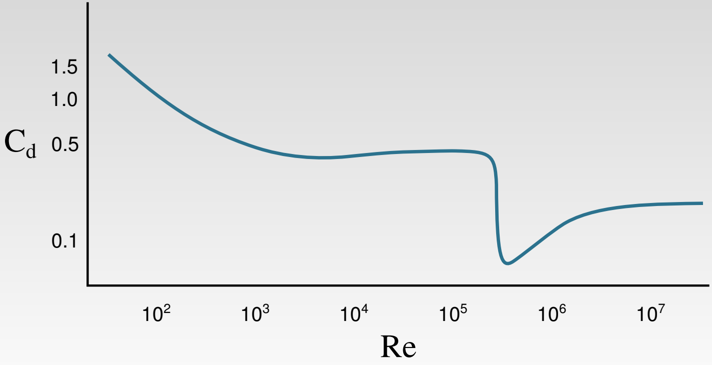
$$\pi_1 = \frac{F}{\rho D^2 v^2} \qquad \pi_2 = \frac{\rho v D}{\mu}$$

π_1 is called *drag coefficient* (C_d), π_2 is called Reynolds *number* (Re).

Buckingham π theorem

Application : drag force

Every cases relating to the drag of a sphere in a viscous fluide are reduced to a *single* curve $C_d = f(Re)$!



Summary

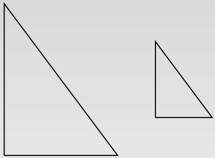
Similitude conditions

Are the measurements on scale models transferable to the prototype?

- Yes, if and only if both configurations check the *similarity conditions*.
- Three conditions :
  - Geometric similarity
  - Kinematic similarity
  - Dynamic similarity

Geometric similarity

- Two geometrical objects are called similar if they both have the same shape. Thus one can be obtained from the other by uniformly scaling.



- The ratio of two homologous distances, connecting two homologous points, has a constant value : this is the geometric similarity scale $L^* = \frac{L_1}{L_2}$.
- Base quantity : L.

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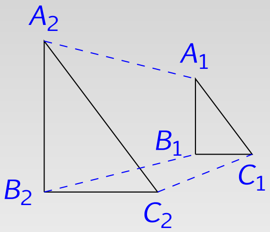

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Geometric similarity

Is it enough ?

Photo L. Shyamal, licence cc-by-sa-2.5

Geometric similarity

Is it enough ?

Kinematic similarity

- Two *geometrically similar* systems are also *cinematically similar* if two homologous particles occupy homologous positions at homologous times.
- In this case, velocity vectors and acceleration vectors at homologous points will also have homologous modules and directions in homologous times.
- Fluide streamlines are then similar.
- Base quantities : L et T.

L^* et T^* are different *in general case* !

Dynamic similarity

- Two systems, *geometrically and cinematically similar*, are also *dynamically similar* if their homologous parts are subjected to homologous force systems, at homologous times.
- The similarity is complete if the ratios between all[1] homologous forces are equal.
- **Ratio of inertia forces :** If there is geometric and kinematic similarity, there is also dynamic similarity for inertial forces, *if the mass distribution is similar* ($\rho^* = \frac{\rho_1}{\rho_2} = \text{cste}$).
- Base quantities : L, M & T.

[1] viscosity, pressure, gravity, surface tension, elasticity...

Dynamic similarity

Ratio of inertial forces to pressure forces (Euler number):

$$\frac{ma}{pS} = \frac{(\rho L^3)(L/T^2)}{pL^2} = \frac{(\rho L^2)(L^2/T^2)}{pL^2} = \frac{\rho L^2 v^2}{pL^2} = \frac{\rho v^2}{p} = \mathrm{Eu}$$

Ratio of inertial forces to viscosity forces:

$$\frac{ma}{\mu \frac{\mathrm{d}v}{\mathrm{d}z} L^2} = \frac{\rho L^3 L T^{-2}}{\mu L T^{-1} L^{-1} L^2} = \frac{\rho v L}{\mu} = \mathrm{Re}$$

Viscous forces dominate at low Reynolds numbers, inertial forces dominate at high Reynolds numbers.

Dynamic similarity

Ratio of inertial forces to gravity forces (Froude number):

$$\frac{ma}{mg} = \frac{\rho L^2 v^2}{\rho L^3 g} = \frac{v^2}{Lg} = \mathrm{Fr}^2$$

Ratio of inertial forces to elasticity forces (Mach number):

$$\frac{ma}{ES} = \frac{\rho L^2 v^2}{EL^2} = \frac{\rho v^2}{E} = \mathrm{Ma}^2$$

This has a major role in the flow of compressible fluids.

Ratio of inertial forces to surface tension forces (Weber number):

$$\frac{ma}{\sigma L} = \frac{\rho L^2 v^2}{\sigma L} = \frac{\rho L v^2}{\sigma} = \mathrm{We}$$

Summary

Dimensional homogeneity

- Let's be G, derived from base quantities L, M & T ; its scale ratio is :

$$G^* = \frac{G_1}{G_2} = f(L^*, M^*, T^*)$$

- $f(L^*, M^*, T^*)$ is G's *dimensional formula.*
- If scaling factor G^* is 1, then G is *dimensionless,* i.e *independant of any unit system.*

Similitude of physical variables

- Let's be u, v, unknown measurable values, functions of three measureable values a, b, c : $u = f_1(a, b, c)$ et $v = f_2(a, b, c)$
- f_1 et f_2 *do not* need to be know, as long as they *exist*
- There is similarity, if one can find a^*, b^*, c^*, scaling factors different from 0 and from ∞ so that : $uu^* = f_1(aa^*, bb^*, cc^*)$ et $vv^* = f_2(aa^*, bb^*, cc^*)$
- Scaling factors a^*, b^*, c^* must satisfy some relations, which contitute similarity conditions.
- Relations between unknown factors u^*, v^* and a^*, b^*, c^* are the *results of similitude*.
- This can be obtained without knowledge of f_1 nor f_2.

Summary

Complete similarity

- Dimensional ananlyse of independant variables lead to a law of the following form :

$$f(\pi_1, \pi_2, \ldots \pi_p) = 0$$

- If $p-1$ dimensionless products are identical for two flows, then the last one is identical too.
- Equality of $p-1$ dimensionless products constitutes *complete similitude condition* between two flows.

Partial similarity

- In practice, it is rarely possible to achieve the complete similarity condition. It is rare to obtain the equality of *every* π characterizing the phenomenon.
- One then tries to achieve a limited similarity by neglecting the numbers π whose influence on the phenomenon studied is the weakest. This is an approximation that leads to a certain degree of uncertainty.

Conclusion

The advantages of dimensional analysis:

- *Predict* the form of an equation
- *Check* the validity of a result
- *Simplify* the relationships between many variables
- *Study* complex phenomena with *model* systems